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Fractional Partial Differential Problem of Some Matrix Two-Variables Fractional Functions

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we find arbitrary order fractional partial derivative of two types of matrix two-variables fractional functions. Fractional binomial theorem plays an important role in this article. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, fractional partial derivative, fractional binomial theorem.**

I. INTRODUCTION

Fractional calculus is a research hotspot in recent years. The application of fractional calculus in many fields such as numerical analysis, physics and engineering has aroused great interest [1-16]. In the first half of the 19th century, Abel, Liouville and Riemann correctly introduced fractional integral and derivative in the analysis. However, the use of generalized differential and integral operators became more familiar in the last decades of the 19th century because of the symbolic calculus of Heaviside and the work of mathematicians such as Hadamard, Hardy and Littlewood, M. Riesz, and H. Weyl.

Fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [17-22]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative, we obtain arbitrary order fractional partial derivative of the following two types of matrix two-variables fractional functions:

$$E_\alpha\left[A\cdot\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m}{2p}(-1)^p r^{m-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2p)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right]$$
$$\otimes_\alpha cos_\alpha\left[A\cdot\sum_{q=0}^{\lfloor (m-1)/2\rfloor}\binom{m}{2q+1}(-1)^q r^{m-2q-1}s^{2q+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2q-1)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha(2q+1)}\right],$$

and

$$E_\alpha\left[A\cdot\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m}{2p}(-1)^p r^{m-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2p)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right]$$
$$\otimes_\alpha sin_\alpha\left[A\cdot\sum_{q=0}^{\lfloor (m-1)/2\rfloor}\binom{m}{2q+1}(-1)^q r^{m-2q-1}s^{2q+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2q-1)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha(2q+1)}\right],$$

where $0<\alpha\leq 1$, m is a positive integer, and A is a matrix. Jumarie type of R-L fractional derivative, a new multiplication of fractional analytic functions, and fractional binomial theorem play important roles in this paper. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([23])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer n, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the n-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([24])**:** *If α,β,x_0,C are real numbers and $\beta\geq\alpha>0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{3}$$

Definition 2.3 ([25])**:** Let x,x_0 and a_k be real numbers for all k, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at $x=x_0$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([26])**:** If $0<\alpha\leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}.\end{aligned} \tag{7}$$

Definition 2.5 ([27])**:** Assume that $0<\alpha\leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{8}$$

And the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2k}, \tag{9}$$

and

$$sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2k+1)}. \tag{10}$$

Theorem 2.6 (matrix fractional Euler's formula)([21])**:** *If* $0<\alpha\leq 1$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha)=cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha). \tag{11}$$

Notation 2.7: If the complex number $z=p+iq$, where p,q are real numbers, and $i=\sqrt{-1}$. p, the real part of z, is denoted by $\mathrm{Re}(z)$; q the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

Notation 2.8: If s is a real number, the largest integer less than or equal to s is denoted by $\lfloor s\rfloor$.

Notation 2.9: If r is any real number, p is any positive integer. Define $(r)_p=r(r-1)\cdots(r-p+1)$, and $(r)_0=1$.

Definition 2.10: Let u,v be non-negative integers. For the two-variables fractional function $f_\alpha(x^\alpha,y^\alpha)$, its u-times fractional partial derivative with respect to x^α, v-times fractional partial derivative with respect to y^α, forms a $p+q$-th order fractional partial derivative, and denoted by $\left({}_{y_0}D_y^\alpha\right)^v\left({}_{x_0}D_x^\alpha\right)^u[f_\alpha(x^\alpha,y^\alpha)]$.

Theorem 2.11 (fractional binomial theorem)**:** *If* $0<\alpha\leq 1$, n *is a positive integer and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α-*fractional analytic functions. Then*

$$[f_\alpha(x^\alpha)+g_\alpha(x^\alpha)]^{\otimes_\alpha n}=\sum_{k=0}^{n}\binom{n}{k}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha(n-k)}\otimes_\alpha\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha k}, \tag{12}$$

Where, $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

III. MAIN RESULTS

In this section, we find arbitrary order fractional partial derivative of two types of matrix two-variables fractional functions. At first, two lemmas are needed.

Lemma 3.1: *If* $0<\alpha\leq 1$, r,s *are real numbers, and* m *is a positive integer. Then*

$$\mathrm{Re}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha+is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}\right]=\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m}{2p}(-1)^p r^{m-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2p)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p} \quad\ldots\ldots\ldots\ldots(13)$$

$$\mathrm{Im}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha+is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}\right]=$$

$$\sum_{q=0}^{\lfloor(m-1)/2\rfloor}\binom{m}{2q+1}(-1)^q r^{m-2q-1}s^{2q+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2q-1)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha(2q+1)} \quad\ldots\ldots\ldots(14)$$

Proof : By fractional binomial theorem,

$$\mathrm{Re}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha+is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}\right]$$

$$=\frac{1}{2}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha+is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}+\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha-is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}\right]$$

$$=\frac{1}{2}\left[\sum_{k=0}^{m}\binom{m}{k}\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-k)}\otimes_\alpha\left(is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha k}+\sum_{k=0}^{m}\binom{m}{k}\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-k)}\otimes_\alpha\left(-is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha k}\right]$$

$$=\frac{1}{2}\left[\sum_{k=0}^{m}\binom{m}{k}[1+(-1)^k]\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-k)}\otimes_\alpha\left(is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha k}\right]$$

$$=\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m}{2p}\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2p)}\otimes_\alpha\left(is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}$$

$$=\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m}{2p}(-1)^p r^{m-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-2p)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}.$$

Similarly,

$$\mathrm{Im}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]$$

$$=\frac{1}{2i}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}-\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]$$

$$=\frac{1}{2i}\left[\sum_{k=0}^{m}\binom{m}{k}\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-k)}\otimes_{\alpha}\left(is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}k}-\sum_{k=0}^{m}\binom{m}{k}\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-k)}\otimes_{\alpha}\left(-is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}k}\right]$$

$$=\frac{1}{2i}\left[\sum_{k=0}^{m}\binom{m}{k}[1-(-1)^k]\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-k)}\otimes_{\alpha}\left(is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}k}\right]$$

$$=\frac{1}{i}\left[\sum_{q=0}^{\lfloor(m-1)/2\rfloor}\binom{m}{2q+1}\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-2q-1)}\otimes_{\alpha}\left(is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}(2q+1)}\right]$$

$$=\sum_{q=0}^{\lfloor(m-1)/2\rfloor}\binom{m}{2q+1}(-1)^q r^{m-2q-1}s^{2q+1}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-2q-1)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}(2q+1)}.$$ q.e.d.

Lemma 3.2: *If* $0<\alpha\leq 1$, r,s *are real numbers,* m *is a positive integer, and* A *is a matrix. Then*

$$\mathrm{Re}\left\{E_{\alpha}\left[A\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]\right\}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}A^k\sum_{p=0}^{\lfloor mk/2\rfloor}\binom{mk}{2p}(-1)^p r^{mk-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(mk-2p)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}2p}. \tag{15}$$

$$\mathrm{Im}\left\{E_{\alpha}\left[A\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]\right\}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k!}A^k\sum_{q=0}^{\lfloor(mk-1)/2\rfloor}\binom{mk}{2q+1}(-1)^q r^{mk-2q-1}s^{2q+1}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(mk-2q-1)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}(2q+1)}. \tag{16}$$

Proof $\mathrm{Re}\left\{E_{\alpha}\left[A\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]\right\}$

$$=\mathrm{Re}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}A^k\left(\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right)^{\otimes k}\right\}$$

$$=\mathrm{Re}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}A^k\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}mk}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}A^k\mathrm{Re}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}mk}\right]$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}A^k\sum_{p=0}^{\lfloor mk/2\rfloor}\binom{mk}{2p}(-1)^p r^{mk-2p}s^{2p}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(mk-2p)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}2p}.$$

Similarly,

$$\mathrm{Im}\left\{E_{\alpha}\left[A\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right]\right\}$$

$$=\mathrm{Im}\left\{\sum_{k=1}^{\infty}\frac{1}{k!}A^k\left(\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}m}\right)^{\otimes k}\right\}$$

$$=\mathrm{Im}\left\{\sum_{k=1}^{\infty}\frac{1}{k!}A^k\left(r\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+is\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes_{\alpha}mk}\right\}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} A^k \mathrm{Im}\left[\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha + is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha mk}\right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q r^{mk-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (mk-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)} .$$

q.e.d.

Theorem 3.3: *If* $0 < \alpha \le 1$, *and* u, v *are non-negative integers,* m *is a positive integer, and* A *is a matrix. Then*

$$\left({}_0D_y^\alpha\right)^v \left({}_0D_x^\alpha\right)^u \left[\begin{array}{c} E_\alpha\left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right] \\ \otimes_\alpha cos_\alpha\left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)}\right] \end{array}\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{p=0}^{\lfloor mk/2 \rfloor} \binom{mk}{2p} (-1)^p (mk-2p)_u (2p)_v r^{mk-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (mk-2p-u)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2p-v)} . \quad (17)$$

And

$$\left({}_0D_y^\alpha\right)^v \left({}_0D_x^\alpha\right)^u \left[\begin{array}{c} E_\alpha\left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right] \\ \otimes_\alpha sin_\alpha\left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)}\right] \end{array}\right]$$

=

$\sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q (mk-2q-$

1)u(2q+1)vrmk−2q−1s2q+11Γ(α+1)xα⊗α(mk−2q−1−u)⊗α1Γ(α+1)yα⊗α(2q+1−v).

(18)

Proof: By Lemma 3.1 and Lemma 3.2,

$$E_\alpha\left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right]$$

$$\otimes_\alpha cos_\alpha\left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)}\right]$$

$$= \mathrm{Re}\left\{\begin{array}{c} E_\alpha\left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right] \\ \otimes_\alpha E_\alpha\left[iA \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)}\right] \end{array}\right\}$$

$$= \mathrm{Re}\left\{E_\alpha\left[A\left(r\frac{1}{\Gamma(\alpha+1)}x^\alpha + is\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha m}\right]\right\}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{p=0}^{\lfloor mk/2 \rfloor} \binom{mk}{2p} (-1)^p r^{mk-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (mk-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p} . \quad \ldots\ldots\ldots(19)$$

Therefore,

$$\left({}_0D_y^\alpha\right)^v \left({}_0D_x^\alpha\right)^u \left[\begin{array}{c} E_\alpha\left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right] \\ \otimes_\alpha cos_\alpha\left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha (2q+1)}\right] \end{array}\right]$$

$$= \left({}_0D_y^\alpha\right)^v \left({}_0D_x^\alpha\right)^u \left[\sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{p=0}^{\lfloor mk/2 \rfloor} \binom{mk}{2p} (-1)^p r^{mk-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (mk-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes_\alpha 2p}\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{p=0}^{\lfloor mk/2 \rfloor} \binom{mk}{2p} (-1)^p r^{mk-2p} s^{2p} \left({}_0D_y^{\alpha} \right)^v \left({}_0D_x^{\alpha} \right)^u \left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (mk-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha 2p} \right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{p=0}^{\lfloor mk/2 \rfloor} \binom{mk}{2p} (-1)^p (mk-2p)_u (2p)_v r^{mk-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (mk-2p-u)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2p-v)} .$$

On the other hand,

$$E_\alpha \left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha 2p} \right]$$

$$\otimes_\alpha sin_\alpha \left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} \right]$$

$$= \mathrm{Im} \left\{ \begin{array}{c} E_\alpha \left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha 2p} \right] \\ \otimes_\alpha E_\alpha \left[iA \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} \right] \end{array} \right\}$$

$$= \mathrm{Im} \left\{ E_\alpha \left[A \left(r \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + is \frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha m} \right] \right\}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q r^{mk-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (mk-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} . \quad (20)$$

Thus,

$$\left({}_0D_y^{\alpha} \right)^v \left({}_0D_x^{\alpha} \right)^u \left[\begin{array}{c} E_\alpha \left[A \cdot \sum_{p=0}^{\lfloor m/2 \rfloor} \binom{m}{2p} (-1)^p r^{m-2p} s^{2p} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2p)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha 2p} \right] \\ \otimes_\alpha sin_\alpha \left[A \cdot \sum_{q=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2q+1} (-1)^q r^{m-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (m-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} \right] \end{array} \right]$$

$$= \left({}_0D_y^{\alpha} \right)^v \left({}_0D_x^{\alpha} \right)^u \left[\sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q r^{mk-2q-1} s^{2q+1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (mk-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} \right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q r^{mk-2q-1} s^{2q+1} \left({}_0D_y^{\alpha} \right)^v \left({}_0D_x^{\alpha} \right)^u \left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (mk-2q-1)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)} y^{\alpha} \right)^{\otimes_\alpha (2q+1)} \right]$$

$=$

$\sum_{k=1}^{\infty} \frac{1}{k!} A^k \sum_{q=0}^{\lfloor (mk-1)/2 \rfloor} \binom{mk}{2q+1} (-1)^q (mk-2q-$
1)u(2q+1)vrmk−2q−1s2q+11Γ(α+1)xα⊗α(mk−2q−1−u)⊗α1Γ(α+1)yα⊗α(2q+1−v) .

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative, we obtain arbitrary order fractional partial derivative of two types of matrix two-variables fractional functions. A new multiplication of fractional analytic functions and fractional binomial theorem play important roles in this article. In fact, our results are generalizations of the results in traditional calculus. In the future, we will continue to use our methods to solve the problems in applied mathematics and fractional differential equations.

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